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LIE GROUP STABILITY OF FINITE DIFFERENCE SCHEMES

Emma Hoarau [1], Claire David [2], Pierre Sagaut [2] and Thiên-Hiêp Lê [1]

[1] ONERA, Computational Fluid Dynamics and Aeroacoustics Department (DSNA)
BP 72, 29 avenue de la Division Leclerc
92322 Châtillon Cedex, France
[2] Université Pierre et Marie Curie-Paris 6
Laboratoire de Modélisation en Mécanique, UMR CNRS 7607
Boîte courrier $n^0$162, 4 place Jussieu, 75252 Paris, cedex 05, France

Abstract. Differential equations arising in fluid mechanics are usually derived from the intrinsic properties of mechanical systems, in the form of conservation laws, and bear symmetries, which are not generally preserved by a finite difference approximation, and leading to inaccurate numerical results. This paper develops a method that enables us to build a scheme that preserves those symmetries. The method is based on the concept of the differential approximation. A comparison of numerical performance of the invariant schemes, standard ones and higher order one has been realised for the Burgers equation.

1. **Introduction.** Lie groups were introduced by Sophus Lie in 1870 in order to study the symmetries of differential equations, yielding thus analytical solutions. Literature provides substantial works and applications, [3], [4]. Symmetry groups can be determined by an automatic procedure, but it often turn out to be tedious and induce errors. A large amount of packages using symbolic manipulations of mathematical expressions have been written. We mention here some of those works: Schwartz [17], Vu and Carminati[14], Herod [15], Baumann [16], Cantwell [5].
In this paper we are interesting in the application of the theory of Lie group to numerical analysis.
Finite difference equations used to approximate the solutions of a differential equation generally do not respect the symmetries of the original equation, and can lead to inaccurate numerical results. Various techniques, that enable us to build a scheme preserving the symmetries of the original differential equation, have been studied. One of these techniques consists in constructing an invariant scheme from a given one by applying the method of the moving frame in [7], [8]. Another one consists in constructing an invariant scheme with the help of the discret invariants of its symmetry group [9], [10], [11], [12], [13] and provides the building of symmetry-adapted meshes, in preserving the differential equation symmetries. This technique is based on a direct study of the symmetries of difference equations and lattices.
Yanenko [2] and Shokin [1], proposed to apply the Lie group theory to finite difference equations by means of the differential approximation. Thus, they have set down conditions under which the differential representation of a finite difference scheme preserves the group of continuous symmetries of the original differential equation. They provide a dissipative scheme, which is called invariant scheme. The resulting scheme is independent of any change of the reference frame, and its differential representation is invariant under the symmetries of the original equation. Ames, Postell and Adams [6] have already used the approach of Yanenko and Shokin to present invariant schemes in which terms are added to the original

difference scheme. They showed that, in specific cases, the invariant scheme is as accurate as high order numerical methods.

In this paper, we focus on the last approach. The method is implemented on some standard schemes for the Burgers equation. A comparison is made between the numerical solutions of these schemes and the invariant scheme.

The paper is organized as follows. Definitions and invariance condition for differential equations are provided in section 2. Section 3 recalls the approach of Yanenko and Shokin. Section 4 concentrates on classical schemes. In section 5, we present a method that enables us to build an invariant scheme with respect to an otherwise lost symmetry.

2. **Definitions and invariance condition for differential equations.** A r-parameter Lie group G_r of point transformations in the Euclidean space $\mathcal{E}(x,u)$ can be written under the form:

$$G_r = \{x_i^* = \phi_i(x,u,a);\ u_j^* = \varphi_j(x,u,a),\ i=1,\dots,m;\ j=1,\dots,n\} \tag{1}$$

Consider a system of l^{th}-order differential equations:

$$\mathcal{F}^\lambda\big(x,u,u^{(k_1)},u^{(k_1,k_2)},\dots,u^{(k_1\dots k_l)}\big)=0,\quad \lambda=1,\dots,q \tag{2}$$

Denote by $u^{(k_1\dots k_p)}$ the vector, the components of which are partial derivatives of order p, namely, $u_j^{(k_1\dots k_p)} = \frac{\partial^p u_j}{\partial x_{k_1}\dots\partial x_{k_p}}$ $j=1,\dots,n$ and $k_1,\dots,k_p \in \{1,\dots,m\}$.

Denote by $x=(x_1,\dots,x_m)$ the independent variables, $u=(u_1,\dots,u_n)$ the dependent variables, and $(x_{k_1}\dots x_{k_p})$ a set of elements of the independent variables.

Equation (2) is a subset of the Euclidean space $\mathcal{E}\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_l)}\big)$. In order to take into account the derivative terms involved in the differential equation, the action of the group G_r of transformations in the space $\mathcal{E}\big(x,u)$ needs to be extended to the space of the derivatives of the dependent variables.

Denote by $\widetilde{G}_r^{(l)}$ a r-parameter Lie group of point transformation in the space $\mathcal{E}\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_l)}\big)$ of the independent variables, dependent variables and the derivative of the dependent variables with respect to the independent ones.

The l^{th}-prolongation operator of G_r is:

$$\widetilde{\mathbf{L}}_\alpha^{(l)} = \xi_i^\alpha(x,u)\frac{\partial}{\partial x_i} + \eta_j^\alpha(x,u)\frac{\partial}{\partial u_j} + \sigma_j^{\alpha,(k_1)}\frac{\partial}{\partial {u_j}^{(k_1)}} + \dots + \sigma_j^{\alpha,(k_1\dots k_l)}\frac{\partial}{\partial {u_j}^{(k_1\dots k_l)}}, \tag{3}$$
$$i=1,\dots,m;\ j=1,\dots,n;\ \alpha=1,\dots,r.$$

ξ_i^α, η_j^α, $\sigma_j^{\alpha,(k_1)}$ and $\sigma_j^{\alpha,(k_1\dots k_o)}$ are given by:

$$\xi_i^\alpha = \frac{\partial\phi_i}{\partial a_\alpha}\Big|_{a=0},\qquad \eta_j^\alpha = \frac{\partial\varphi_j}{\partial a_\alpha}\Big|_{a=0},\qquad \sigma_j^{\alpha,(k_1)} = \frac{\mathcal{D}\eta_j^\alpha}{\mathcal{D}x_{k_1}} - \sum_{i=1}^m \frac{\partial u_j}{\partial x_i}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_1}}$$

$$\sigma_j^{\alpha,(k_1\dots k_o)} = \frac{\mathcal{D}\sigma_j^{\alpha,(k_1\dots k_{o-1})}}{\mathcal{D}x_{k_o}} - \sum_{i=1}^m \frac{\partial^o u_j}{\partial x_i\partial x_{k_1}\dots\partial x_{k_{o-1}}}\frac{\mathcal{D}\xi_i^\alpha}{\mathcal{D}x_{k_o}},\quad o=2,\dots,l$$

where: $\frac{\mathcal{D}}{\mathcal{D}x_k} = \frac{\partial}{\partial x_k} + \sum_{j=1}^n \frac{\partial u_j}{\partial x_k}\frac{\partial}{\partial u_j}$

The system of l^{th}-order differential equations is invariant under the group $\widetilde{G}_r^{(l)}$ if and only if:

$$\widetilde{\mathbf{L}}_\alpha^{(l)}\mathcal{F}^\lambda\Big|_{\mathcal{F}^\lambda=0} = 0,\quad \alpha=1,\dots,r;\ \lambda=1,\dots,q \tag{4}$$

3. **Lie group for the differential approximation.** The finite difference scheme, which approximates the differential system (2), can be written as:

$$\Lambda^\lambda(x,u,h,Tu)=0, \quad \lambda=1,\dots,q \tag{5}$$

where $h=(h_1,h_2,\dots,h_m)$ denotes the space step vector, and $T=(T_1,T_2,\dots,T_m)$ the shift-operator along the axis of the independent variables, defined by:

$$T_i[u](x_1,x_2,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m)=u(x_1,x_2,\dots,x_{i-1},x_i+h_i,x_{i+1},\dots,x_m). \tag{6}$$

Definition 1. The differential equation:

$$\begin{aligned}\mathcal{P}^\lambda\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})}\big) &= \mathcal{F}^\lambda\big(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_l)}\big)\\ &+\sum_{\beta=1}^{s}\sum_{i=1}^{m}(h_i)^{l_\beta}\mathcal{R}_i^\lambda(x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'}{}_{\lambda,i})}),\\ &\lambda=1,\dots,q;\ l'=max_{(\lambda,i)}l'{}_{\lambda,i}\end{aligned} \tag{7}$$

is called the s^{th}-order differential approximation of the finite difference scheme (5). In the specific case $s=1$, the above equation is called the first differential approximation.

Equation (7) is obtained from equation (5) by applying Taylor series expansion to the components of Tu about the point $x=(x_1,\ \dots,\ x_m)$ and truncating the expansion to a given finite order. Denote by G'_r a group of transformations in the space $\mathcal{E}(x,u,h)$:

$$G'_r=\{x_i^*=\phi_i(x,u,a);\ u_j^*=\varphi_j(x,u,a); h_i^*=\psi_i(x,u,h,a),\ i=1,\dots,m;\ j=1,\dots,n\} \tag{8}$$

by $\mathbf{L}_\alpha{}'$ the basis infinitesimal operator of G'_r:

$$\mathbf{L}_\alpha{}'=\xi_i^\alpha(x,u)\frac{\partial}{\partial x_i}+\eta_j^\alpha(x,u)\frac{\partial}{\partial u_j}+\zeta_i^\alpha(x,u,h)\frac{\partial}{\partial h_i}, \quad \alpha=1,\dots,r \tag{9}$$

where

$$\zeta_i^\alpha=\frac{\partial\psi_i}{\partial a_\alpha}\Big|_{a=0}, \quad \alpha=1,\dots,r \tag{10}$$

and by $\widetilde{G}_r^{(l')}$ a group of transformation in the space $\mathcal{E}(x,u,h,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})})$. The l'^{th}-prolongation operator of G'_r, $\widetilde{\mathbf{L}}_\alpha^{(l')}$ can be written as:

$$\widetilde{\mathbf{L}}_\alpha^{(l')}=\mathbf{L}_\alpha{}'+\sum_{j=1}^{n}\sum_{p=1}^{l'}\sigma_j^{\alpha,(k_1\dots k_p)}\frac{\partial}{\partial u_j^{(k_1\dots k_p)}} \tag{11}$$

Theorem 1. *The differential approximation (7) is invariant under the group $\widetilde{G}_r^{(l')}$ if and only if*

$$\widetilde{\mathbf{L}}_\alpha^{(l')}\mathcal{P}^\lambda\big((x,u,u^{(k_1)},\dots,u^{(k_1\dots k_{l'})}\big)\Big|_{\mathcal{P}^\lambda=0}=0, \quad \alpha=1,\dots,r;\ \lambda=1,\dots,q \tag{12}$$

or

$$\Big[\widetilde{\mathbf{L}}_\alpha^{(l)}\mathcal{F}^\lambda+\widetilde{\mathbf{L}}_\alpha^{(l')}\Big(\sum_{\beta=1}^{s}\sum_{i=1}^{m}(h_i)^{l_\beta}\mathcal{R}_i^\lambda\Big)\Big]\Big|_{\mathcal{P}^\lambda=0}=0, \quad \alpha=1,\dots,r;\ \lambda=1,\dots,q \tag{13}$$

Theorem 1 provides the equations which enable us to obtain the symmetry groups of the differential approximation. The unknowns are the infinitesimal functions η_j^α, ξ_i^α and ζ_i^α, $i=1,\dots,m$; $j=1,\dots,n$. The infinitesimals $\sigma_j^{\alpha,(k_1,\dots,k_o)}$, $j=1,\dots,n$, are functions of the partial derivatives of η_j^α and ξ_i^α.

Equation (13) is simplified by means of the condition (7). They lead to an overdetermined system of differential equations, the unknowns of which are the infinitesimal functions.

4. **The specific case of the Burgers equation.**

4.1. **Symmetries of the Burgers equation.**
The Burgers equation can be written as:

$$\mathcal{F}(x,t,u,\nu,u_x,u_t,u_{xx}) = u_t + u\ u_x - \nu\ u_{xx} = 0 \tag{14}$$

where $\nu \geq 0$ is the dynamic viscosity.
Denote by G a group of transformations of the Burgers equation in the space $\mathcal{E}(x,t,u,\nu)$ of the independent variables (x,t), the dependent variable u, and the viscosity ν. The viscosity is taken as a symmetry variable in order to enable us to take into account variations of the Reynolds number.
G is a set of transformations acting smoothly on the space $\mathcal{E}(x,t,u,\nu)$.
The six-dimensional Lie algebra of the group G is generated by the following operators:

$$\begin{aligned}
&\mathbf{L}_1 = \frac{\partial}{\partial x},\ \mathbf{L}_2 = \frac{\partial}{\partial t},\ \mathbf{L}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\\
&\mathbf{L}_4 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (-ut+x)\frac{\partial}{\partial u},\ \mathbf{L}_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ \mathbf{L}_6 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \nu\frac{\partial}{\partial \nu}
\end{aligned} \tag{15}$$

which respectively correspond to:

- the space translation : $(x,t,u,\nu) \longmapsto (x+\epsilon_1, t, u, \nu)$;
- the time translation : $(x,t,u,\nu) \longmapsto (x, t+\epsilon_2, u, \nu)$;
- the dilatation : $(x,t,u,\nu) \longmapsto (\epsilon_3 x, \epsilon_3^2 t, \epsilon_3^{-1} u, \nu)$;
- the projective transformation : $(x,t,u,\nu) \longmapsto \Big(\frac{x}{1-\epsilon_4 t}, \frac{t}{1-\epsilon_4 t}, x\epsilon_4 + u(1-\epsilon_4 t), \nu\Big)$;
- the Galilean transformation : $(x,t,u,\nu) \longmapsto (x+\epsilon_5\ t, t, u+\epsilon_5, \nu)$;
- the dilatation : $(x,t,u,\nu) \longmapsto (x, \epsilon_6^{-1} t, \epsilon_6 u, \epsilon_6 \nu)$.

$(\epsilon_i)_{i=1,\ldots,6}$ are constants.

4.2. **Symmetries of first differential approximations.**
Denote by h the mesh size, τ the time step, N_x the number of mesh points, N_t the number of time steps, and $u_i^n,\ i \in \{0,\ldots,N_t\},\ n \in \{0,\ldots,N_x\}$ the discrete approximation of $u(ih, n\tau)$.
In order to shorten the size of the finite difference scheme expressions, we use the following notations introduced by Hildebrand in [18]:

$$\begin{aligned}
\delta(u_i^n) &= \frac{u^n_{i+\frac12} - u^n_{i-\frac12}}{h}, \qquad \mu(u_i^n) = \frac{u^n_{i+\frac12} + u^n_{i-\frac12}}{2}\\
\delta^+(u_i^n) &= \frac{u^n_{i+1} - u^n_i}{h}, \qquad \delta^-(u_i^n) = \frac{u^n_i - u^n_{i-1}}{h}, \qquad E^\alpha u_i^n = u^n_{i+\alpha}
\end{aligned}$$

The Burgers equation can be discretized by means of:

- **the FTCS (forward-time and centered-space) scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}(\frac{u^2}{2})_i^n - \nu\frac{\delta^2}{h^2}u_i^n = 0$$

- **the Lax-Wendroff scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}(\frac{u^2}{2})_i^n - \nu\frac{\delta^2}{h^2}u_i^n + A_i^n = 0$$

  where:

$$\begin{aligned}
A_i^n = &- \frac{\tau}{2h^2}\Big[E^{\frac12}u_i^n\ \delta^+(\frac{u^2}{2})_i^n - E^{-\frac12}u_i^n\ \delta^-(\frac{u^2}{2})_i^n\Big] - \frac{\nu^2\tau}{2}\Big[\frac{\delta^4}{h^4}u_i^n\Big]\\
&+ \frac{\nu\tau}{2h^3}\Big[E^{\frac12}u_i^n\ \delta^2(E^{\frac12}u_i^n) - E^{-\frac12}u_i^n\ \delta^2(E^{-\frac12}u_i^n)\Big] + \frac{\nu\tau}{2}\Big[\frac{\mu\delta^3}{h^3}(\frac{u^2}{2})_i^n\Big]
\end{aligned}$$

- **the Crank-Nicolson scheme**:

$$\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{\mu\delta}{h}\Big[(\frac{u^2}{2})_i^{n+1} + (\frac{u^2}{2})_i^n\Big] - \nu\frac{\delta^2}{h^2}[u_i^{n+1} + u_i^n] = 0$$

Linear stability properties and the related orders of approximation are displayed in Table 1 (where $CFL = \frac{a\tau}{h}$, $S = \frac{\nu\tau}{h^2}$ and $S^* = \big(\nu + \frac{ahCFL}{2}\big)\frac{\tau}{h^2}$).

Scheme	**Stablility condition**	**Error**
FTCS	$S \leq \frac{1}{2}$, $CFL \leq 1$	$\mathcal{O}(\tau, h^2)$
Lax-Wendroff	$S^* \leq \frac{1}{2}$, $CFL \leq 1$	$\mathcal{O}(\tau^2, h^2)$
Crank-Nicolson	unconditional stability	$\mathcal{O}(\tau^2, h^2)$

TABLE 1. Table of finite difference schemes

Consider ${u_i}^n$ as a function of the time step τ, and of the mesh size h, expand it at a given order by means of its Taylor series, and neglect the $o(\tau^\alpha)$ and $o(h^\beta)$ terms, where α and β depend on the order of the schemes. This yields the differential representation of the finite difference equation.
The following differential representations are obtained:

- **for the FTCS scheme:**
$$u_t + \frac{1}{2}(u^2)_x - \nu\ u_{xx} + \frac{\tau}{2}g_2 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$
- **for the Lax-Wendroff scheme:**
$$u_t + \frac{1}{2}(u^2)_x - \nu\ u_{xx} + \frac{\tau^2}{6}g_3 + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$
- **for the Crank-Nicolson scheme:**
$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} + \tau^2\Big(\frac{g_3}{6} + \frac{1}{4}(g_1^2 + u g_2)_x - \frac{\nu}{4}(g_2)_{xx}\Big) + \frac{h^2}{12}(u^2)_{xxx} - \frac{\nu h^2}{12}u_{xxxx} = 0$$

where $g_1 = -\big(\frac{u^2}{2}\big)_x + \nu u_{xx}$, $g_2 = \big(-g_1 u\big)_x + \nu\big(g_1\big)_{xx}$, $g_3 = \big(-g_2 u - g_1^2\big)_x + \nu\big(g_2\big)_{xx}$

Denote by G' the group of transformations of a first differential approximation in the space $\mathcal{E}(x, t, u, h, \tau, \nu)$ of the independent variables (x, t) and the dependent variable u, the step size variables (h, τ) and the viscosity ν.
The l'^{th}-prolongation of G' can be written as:

$$\widetilde{\mathbf{L}}_\alpha^{\prime(l')} = \xi_1^\alpha \frac{\partial}{\partial x} + \xi_2^\alpha \frac{\partial}{\partial t} + \eta^\alpha \frac{\partial}{\partial u} + \sum_{p=1}^{l'} \sigma_j^{\alpha,(k_1 \ldots k_p)} \frac{\partial}{\partial u_j^{(k_1 \ldots k_p)}} + \zeta_1^\alpha \frac{\partial}{\partial h} + \zeta_2^\alpha \frac{\partial}{\partial \tau} + \theta^\alpha \frac{\partial}{\partial \nu} \quad (16)$$

where l' has been defined in **definition 1**.
Theorem 1 enables us to obtain the necessary and sufficient condition of invariance of the first differential approximation $\mathcal{P}$:

$$\widetilde{\mathbf{L}}_\alpha^{\prime(l')}\mathcal{P}\Big|_{\mathcal{P}=0} = 0 \quad (17)$$

Theorem 1 is applied to the differential representations of the above schemes.
The resolution of the determining equations of each first differential approximation yields the 4-parameter group:

$$\begin{aligned} &\xi_1^\alpha = a + b\ x, \quad \xi_2^\alpha = c + (2b - d)\ t, \quad \eta^\alpha = (-b + d)\ u \\ &\zeta_1^\alpha = b\ h, \quad \zeta_2^\alpha = (2b - d)\ \tau, \quad \theta^\alpha = e\nu \end{aligned} \quad (18)$$

The 4-dimensional Lie algebra of G' is generated by:

$$\begin{aligned} &\mathbf{L}_1 = \frac{\partial}{\partial x}, \quad \mathbf{L}_2 = \frac{\partial}{\partial t}, \quad \mathbf{L}'_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + h\frac{\partial}{\partial h} + 2\tau\frac{\partial}{\partial \tau} \\ &\mathbf{L}'_4 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} - \tau\frac{\partial}{\partial \tau} + \nu\frac{\partial}{\partial \nu} \end{aligned} \quad (19)$$

These operators are respectively related to:

- the space translation : $(x,t,u,h,\tau,\nu) \longmapsto (x+\epsilon_1,t,u,h,\tau,\nu)$;
- the time translation : $(x,t,u,h,\tau,\nu) \longmapsto (x,t+\epsilon_2,u,h,\tau,\nu)$;
- the dilatation : $(x,t,u,h,\tau,\nu) \longmapsto (\epsilon_3 x,\epsilon_3^2 t,\epsilon_3^{-1}u,\epsilon_3 h,\epsilon_3^2\tau,\nu)$;
- the dilatation : $(x,t,u,h,\tau,\nu) \longmapsto (x,\epsilon_4^{-1}t,\epsilon_4 u,h,\epsilon_4^{-1}\tau,\epsilon_4\nu)$;

where $(\epsilon_i)_{i=1,\ldots,4}$ are constants.
The above finite difference equations are preserved by the space translation, the time translation and both dilatations.
Approximating the Burgers equation by the above finite difference equations results in the loss of the projective and Galilean transformations.

5. **The invariant scheme.**

5.1. **Invariant scheme construction.** An invariant scheme is constructed in such a way that the related differential approximation preserves the symmetries of the Burgers equation. We propose to approximate the Burgers equation by the following finite difference scheme:

$$\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{1}{h}\big(\mu\delta-\frac{\mu\delta^3}{6}\big)\big(\frac{u^2}{2}\big)_i^n-\nu\frac{1}{h^2}\big(\delta^2-\frac{\delta^4}{12}\big)(u_i^n)-\Big(\Omega^n_{i+\frac{1}{2}}\delta^+-\Omega^n_{i-\frac{1}{2}}\delta^-\Big)u_i^n=0 \qquad (20)$$

where $\Omega_i^n=\Omega(x_i,t_n,u_i^n)$ is defined next so that the related differential representation is preserved by the symmetries of the Burgers equation. The scheme has second-order accuracy in space and first-order accuracy in time. The derivatives $\left(\frac{u^2}{2}\right)_x$ and u_{xx} are approximated by fourth order accuracy difference expressions:

$$\big(\frac{\mu\delta}{h}-\frac{\mu\delta^3}{6h}\big)(u_i^n)=\big(u_x-\frac{h^4}{30}u_{5x}\big)_i^n+\mathcal{O}(h^6),\ \big(\frac{\delta^2}{h^2}-\frac{\delta^4}{12h^2}\big)(u_i^n)=\big(u_{xx}-\frac{h^4}{90}u_{6x}\big)_i^n+\mathcal{O}(h^6) \quad (21)$$

The truncation error of the difference scheme (20) can be written as:

$$\epsilon \quad = \quad \frac{\tau}{2}u_{tt}-h^2\Big(\Omega u_x\Big)_x+\mathcal{O}(\tau^2)+\mathcal{O}(h^4)$$

u_{tt} is replaced by an expression involving partial derivatives with respect to x, by using the Burgers equation:

$$u_{tt}=(u^2u_x)_x-\nu(uu_{xx})_x-\nu\big(\frac{u^2}{2}\big)_{xxx}+\nu^2u_{xxxx} \qquad (22)$$

Replacing the previous expression in the truncation error leads to:

$$\epsilon \quad = \quad \Big(Cu_x\Big)_x-\frac{\nu\tau}{2}\Big(uu_{xx}\Big)_x-\frac{\nu\tau}{2}\big(\frac{u^2}{2}\big)_{xxx}+\frac{\nu^2\tau}{2}u_{xxxx}+\mathcal{O}(\tau^2)+\mathcal{O}(h^4)$$

where $C=\dfrac{\tau}{2}u^2-h^2\Omega$.
It is convenient for the calculation of C that the truncation error is reduced to:

$$\epsilon \quad = \quad \Big(Cu_x\Big)_x+\mathcal{O}(\tau^2)+\mathcal{O}(h^4)$$

The related finite difference scheme is the following first order accuracy in time and second order accuracy in space:

$$\begin{aligned}&\frac{u_i^{n+1}-u_i^n}{\tau}+\frac{1}{h}\big(\mu\delta-\frac{\mu\delta^3}{6}\big)\big(\frac{u^2}{2}\big)_i^n-\nu\frac{1}{h^2}\big(\delta^2-\frac{\delta^4}{12}\big)(u_i^n)-\Big(\Omega^n_{i+\frac{1}{2}}\Delta_1-\Omega^n_{i-\frac{1}{2}}\Delta_{-1}\Big)u_i^n\\&\qquad+\frac{\nu\tau}{2}\Big(u^n_{i+\frac{1}{2}}\frac{\mu\delta^2}{h^2}(u^n_{i+\frac{1}{2}})-u^n_{i-\frac{1}{2}}\frac{\mu\delta^2}{h^2}(u^n_{i-\frac{1}{2}})\Big)-\frac{\nu^2\tau}{2}\frac{\delta^4}{h^4}u_i^n+\frac{\nu\tau}{2}\frac{\mu\delta^3}{h^3}\big(\frac{u^2}{2}\big)_i^n=0\end{aligned} \qquad (23)$$

and the differential approximation can be written as:

$$\mathcal{P}(x,t,u,\nu,u_x,u_t,u_{xx})=u_t+u\ u_x-\nu\ u_{xx}+(Cu_x)_x=0 \qquad (24)$$

The von Neumann stability analysis of scheme (23) under a linearized form provides the following necessary conditions for S, CFL and $\Omega_\tau=\Omega\tau$:

$$CFL^2-2S-2\Omega_\tau\leq 0,\qquad 0\leq\frac{4S}{3}-2S^2+\Omega_\tau\leq\frac{1}{2} \qquad (25)$$

If Ω takes is sufficiently close to zero, these conditions become then sufficient for the linear formulation.

5.2. **Calculation of the artificial viscosity term.** Here we describe the method for determining the artificial viscosity term $(Cu_x)_x$, which is constructed in such a way that the differential approximation (24) is preserved by the symmetries of the Burgers equation. C is a function of the variables (x, t, u, τ, h), and also depends on the partial derivatives of u with respect to x: u_x and u_{xx}. $C = C(x, t, h, \tau, u, u_x, u_{xx})$. The necessary and sufficient condition for the differential approximation to be an invariant of the Burgers equation symmetry group is:

$$\widetilde{\mathbf{L}}_\alpha^{(2)}(u_t + u\ u_x - \nu\ u_{xx})\Big|_{\mathcal{P}=0} + \widetilde{\mathbf{L}}_\alpha^{(3)}((Cu_x)_x)\Big|_{\mathcal{P}=0} = 0 \tag{26}$$

Equation (26) provides the determining equations of the symmetry group of equation (24). The determining equations involve partial derivatives of the unknown function C and partial derivatives of the infinitesimal functions of $G^{'}$, which is the symmetry group of the differential representation of the invariant scheme.
The infinitesimal functions of $G^{'}$ have the following expressions:

$$\begin{aligned} &\xi_1^\alpha = a + b\ x + c\ t + d\ tx, \qquad \xi_2^\alpha = e + d\ t^2 + (2b - f)\ t, \qquad \zeta_1^\alpha = b\ h, \\ &\zeta_2^\alpha = (2b - f)\ \tau, \quad \eta^\alpha = c + d\ x + (-b - d\ t + f)\ u, \quad \theta^\alpha = f\nu \end{aligned} \tag{27}$$

The determining equation with respect to the unknown function C is simplified in using the infinitesimal functions of each subgroup of $G^{'}$.
The determining equations of each subgroup of $G^{'}$ provides the following linear partial differential equations and the expressions for C:

- the space translation $\frac{\partial}{\partial x}C = 0 \Rightarrow C = C_1(t, h, \tau, u, u_x, u_{xx})$;
- the time translation $\frac{\partial}{\partial t}C = 0 \Rightarrow C = C_2(x, h, \tau, u, u_x, u_{xx})$;
- the dilatation $x\frac{\partial}{\partial x}C + 2t\frac{\partial}{\partial t}C - u\frac{\partial}{\partial u}C + h\frac{\partial}{\partial h}C + 2\tau\frac{\partial}{\partial \tau}C = 0 \Rightarrow C = C_3(\frac{t}{x^2}, \frac{h}{x}, ux, \frac{\tau}{x^2}, u_x, u_{xx})$;
- the projective transformation $\frac{\partial}{\partial x}C = 0$, $\frac{\partial}{\partial u}C = 0$, $\frac{\partial}{\partial u_{xx}}C = 0$, $t^2\frac{\partial}{\partial t}C + 2\frac{\partial}{\partial u_x}C = 0$ $\Rightarrow C = C_4(h, \tau, \frac{2+tu_x}{t})$;
- the Galilean transformation $\frac{\partial}{\partial u}C + t\frac{\partial}{\partial x}C \Rightarrow C = C_5(\frac{ut-x}{t}, t, h, \tau, u, u_x, u_{xx})$;
- the dilatation $-t\frac{\partial}{\partial t}C + u\frac{\partial}{\partial u}C + \nu\frac{\partial}{\partial \nu}C - \tau\frac{\partial}{\partial \tau}C = 0 \Rightarrow C = \frac{1}{t}C_6(x, h, \frac{\tau}{t})$.

5.3. **Numerical application.** The numerical resolution of the Burgers equation has been implemented for scheme (23), the standard schemes (cf. section 4.2) and a scheme with second-order accuracy in time and fourth-order accuracy in space, which is obtained from the invariant scheme when $C = 0$. The solutions are calculated in the reference frame $(F1)$ and in the one $(F2)$ resulting from the Galilean transformations $(x, t, u, \nu) \longmapsto (x + t, t, u + 1, \nu)$. The artificial viscosity has the following expression:

$$C = -0.01t(tu - x)^2(u_x)^2 \tag{28}$$

$\Omega = \dfrac{1}{h^2}(\dfrac{\tau}{2}u^2 - C)$ is in a sufficiently small neighborhood of zero that we have the sufficiency of conditions (25) for the linear formulation.
The problem consists in solving the following differential system:

$$\begin{gathered} u_t + uu_x - \nu u_{xx} = 0,\ x \in [0, 40],\ t \in [0, 20] \\ u(x, 0) = f(x),\ u(0, t) = g(t),\ u(40, t) = h(t) \end{gathered}$$

The initial and boundary conditions, f, h, and g are provided by an exact solution of the Burgers equation:

$$u(x, t) = \frac{(x - 2t)/(t + 0.1)}{1 + \nu^2\sqrt{t + 0.1}\exp\left((x - 2t)^2/(4\nu(t + 0.1))\right)} + 2 \tag{29}$$

Figures 1, 3 and 5 show the time evolution of the L^2-norm of the error for the considered schemes, for specific values of the CFL number and the mesh Reynolds

number Re_h. Figures 2, 4 and 6 display the variations, as functions of the space variable, of the numerical solutions of the considered schemes for the specific value $t = 5$. In each frame, the numerical solutions are compared to the exact one.
The error analysis of the invariant scheme in the reference frame through the features of the truncation error and the graphical representation of the norms of the error (cf. Figures 1, 3 and 5) allows to say that the invariant scheme is dissipative and slightly dispersive.
The presence of the dissipative term $(Cu_x)_x$ in the differential representation of the invariant scheme and the presence of the higher order error terms involving the even-order derivative u_{6x} (cf. Equation (21)) show that the scheme produces numerical damping. Particularly, the amplitudes are not correctly represented for high frequencies, since the solution is subjected to rather rough variation during the first iterations. The dissipation is stronger for $Re_h = 2$, $CFL = 0.08$ in the reference frame (see Figure 4). Moreover, the presence of higher order error terms involving the odd-order derivative u_{5x} corresponds to a phase error.
The non-invariant schemes are more altered by the change of the frame than the invariant one. Moreover, the invariant scheme appears to be as accurate as the higher order one in the frame (F2).

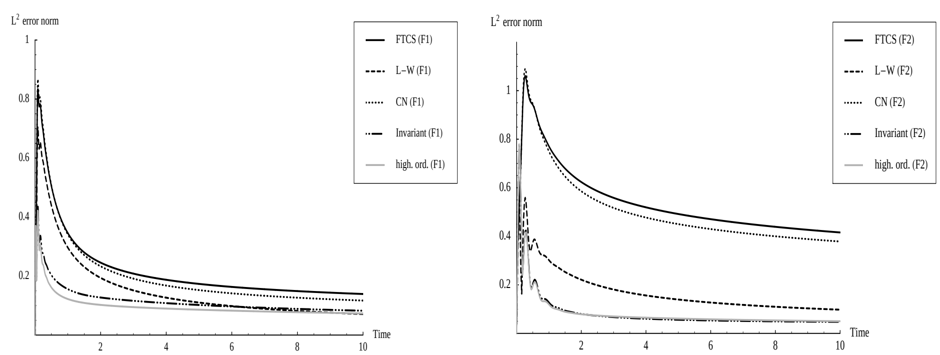

FIGURE 1. Evolution of the error L^2-norm in (F1) and (F2). $Re_h = 2$, $CFL = 0.04$

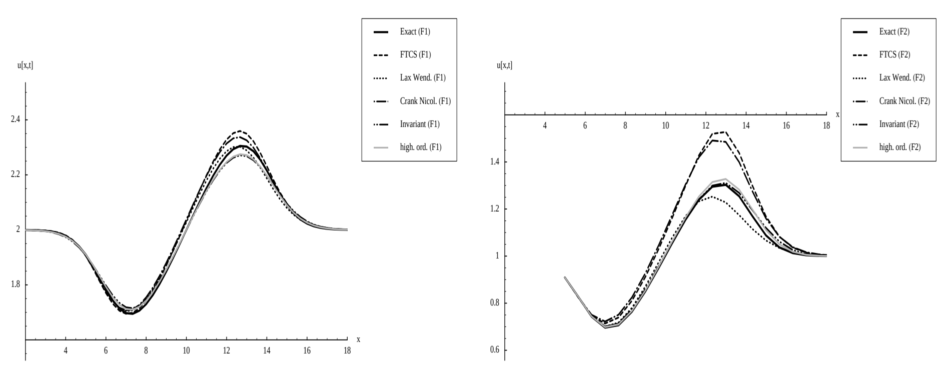

FIGURE 2. Space variation of the numerical solutions of the schemes and the exact solution in $(F1)$ and $(F2)$. $Re_h = 2$, $CFL = 0.04$

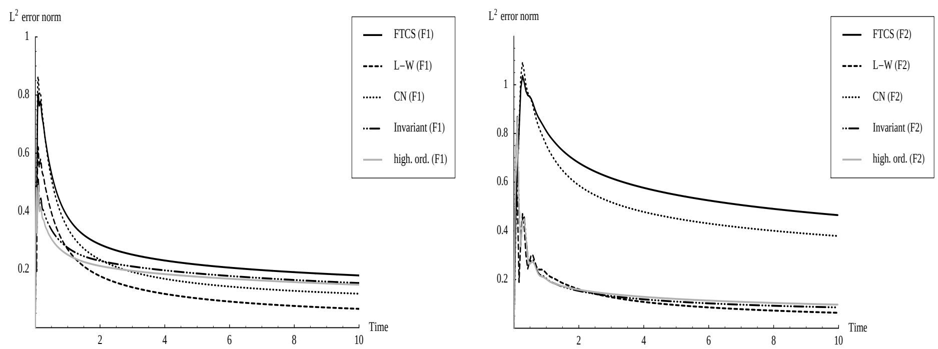

FIGURE 3. Evolution of the error L^2-norm in (F1) and (F2). $Re_h = 2$, $CFL = 0.08$

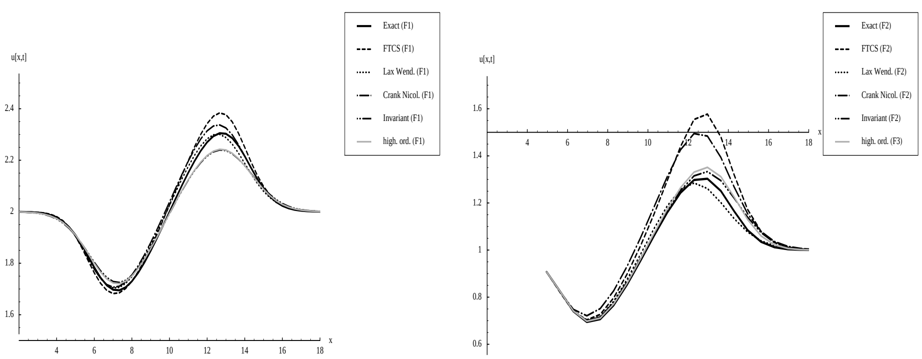

FIGURE 4. Space variation of the numerical solutions of the schemes and the exact solution in $(F1)$ and $(F2)$. $Re_h = 2$, $CFL = 0.08$

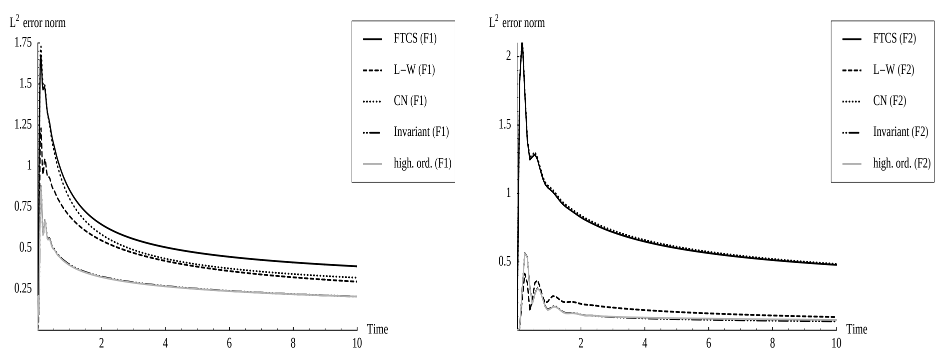

FIGURE 5. Evolution of the error L^2-norm in (F1) and (F2). $Re_h = 3$, $CFL = 0.08$

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E-mail address: `emma.hoarau@onera.fr`

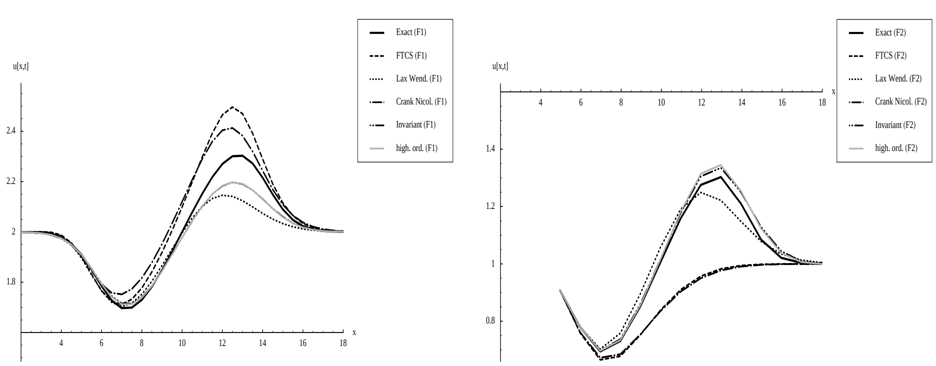

FIGURE 6. Space variation of the schemes numerical and of the exact solution in $(F1)$ and $(F2)$. $Re_h = 3$, $CFL = 0.08$